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Concept Maps to Assess Student Teachers' Understanding of Mathematical Proof

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Abstract: This study reports the analysis of concept maps dealing with "Mathematical Proof," as generated by a group of student teachers. The researcher examined the type of concept map generated, the number of key terms utilized in the construction of the map, and the multiplicity of relationships indicated among those key terms/concepts. The conceptual understanding represented within the concept map was then mapped onto Balacheff's (1988) taxonomy of proofs. The lack of sophistication in the concept maps produced may point towards limitations in student teachers' understanding of mathematical proof. Since teacher's conceptions of proof inevitably influences both the role and nature of the instruction of mathematical proof within a mathematics classroom, limited knowledge in this core area of mathematics may typically prompts feelings of uncertainty and a lack of confidence when it comes to teaching this concept.

Key words: Mathematical proof; Student teacher; Conceptions; Concept map; Balacheff's taxonomy of proofs

Introduction

Assessment is defined as the process of gathering information so as to monitor student progress and make sound instructional decisions. However, it is not an easy task to assess student understanding of mathematical concepts. Mathematics educators have long used paper and pencil tests as tools to assess learning. However, the need for a better way to represent learners' conceptual understanding has led to the development of concept maps as an alternative assessment tool (Novak & Canas, 2006). Within the realm of mathematics education, some researchers (Schimittau, 2004; Afamasa-Fuatai, 2004a, 2004b) specifically advocate the use of concept maps to assess mathematical learning.

A concept map is an explicit, graphical representation of knowledge. Concept maps can *effectively* map what is *inside* the mind to the *outside* (Tergan, 1988) and reveal those conceptual understandings that are not generally identifiable by other assessment tools (Hasemann & Mansfield, 1995). They provide the student with a different mean of demonstrating understanding, and the assessor with an additional opportunity to witness how the student connects ideas and groups or organizes information. In other words, concept maps effectively reveal the overall integrated knowledge of the learner.

The theoretical foundation of concept mapping derives from Ausbel's theory of learning. This theory posits that meaningful learning takes place by assimilating new concepts into existing conceptual frameworks held by the learner (Ausbel, 1963; 1968; Ausbel, Novak & Hanesian, 1978). Learners who are asked to draw a concept map must choose visual symbols and/or details to represent concepts and to clarify the nature and relationships among these concepts. Details and connections between concepts can be added in any order. Generally maps are drawn with concepts contained in oval shapes and words noted on the lines connecting concepts/shapes (see Figure 1).

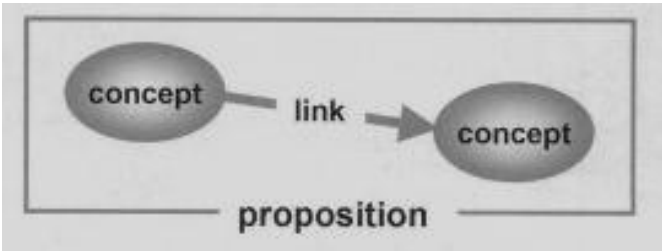

Figure 1. The basic element of a concept map is a proposition, consisting of two concepts connected by a link that shows the relationship between them.

(Source: Rebich & Gautier, 2005, p. 358)

Figure 1.Concepts and Link in a Concept Map

Mathematical Proof and School Curriculum

"Mathematical proof" has been regarded as one of the most distinguishing characteristics of the discipline of mathematics since the nineteenth century (Davis & Hersh, 1981). Mathematician Michael Atiyah identifies proof as "the glue that holds mathematics together" (as cited in Dunham, 1994, p. 15). Wu (1996) considers "proof as the guts of mathematics" (p. 222) and reminds us that mathematics courses are where the students get their training in logical reasoning. He continues that it is through proofs that students learn how to "distinguish between what is true and what only *seems* to be true" (p. 224). He also notes that "any one who wants to know what mathematics is about must therefore learn how to write down a proof or at least understand what a proof is" (p. 222). Hence, its role in school mathematics is quite significant. Despite its perceived importance, however, it tends to have little meaning for students. Knuth (1999, 2002a, 2002b) notes that, in North American schools, because Euclidean geometry is the sole vehicle by which high school students are introduced to mathematical proof, students are rarely able to identify the objectives or functions of mathematical proof. Given this narrow application, it is not surprising that students come to perceive mathematical proof as a formal and meaningless exercise (Chazan, 1993; Coe & Ruthven, 1994; Healy & Hoyles, 2000; Hadas, Hershkowitz & Schwarz, 2000; Weber, 2001).

After its elevation to a standard in the curriculum document *Principles and standards for school mathematics* (2000), published by the National Council of Teachers of Mathematics (NCTM), mathematical proof became one of the most *talked about* topics in mathematics education. NCTM (2000) reminds us that learning mathematics requires far more than simply solving exercises by working with symbols, performing desired calculations and doing some proofs. It is fundamentally about "developing a mathematical view point", "mathematical reasoning", "communicating mathematically", "making connections in mathematics", and building "connections" with other disciplines and [among mathematical] experiences in mathematics. Since learning mathematics involves discovery, "proof and reasoning" are powerful ways of developing insights, making connections, and communicating mathematically. NCTM also underlines the fact that being "able to reason is essential to understanding" mathematics. (p. 56). This suggests that proficiency in mathematical proof and reasoning is an integral part of mathematics.

The role of the teacher is critical in this aspect. As the NCTM emphasizes, "[s]tudents learn mathematics through the experiences that teacher provide" (2000, p. 16). NCTM underscores the fact that teaching shapes students' understanding of mathematics, their ability to use it to solve problems, and their confidence in and attitude towards mathematics. Many researchers have pointed out that teachers' knowledge and belief play a critical role in successfully enacting classroom practices (Fennema & Franke, 1992; Thompson, 1984).

The teaching of mathematical proof places significant demands on the subject-matter knowledge and the pedagogical knowledge of secondary mathematics teachers (Jones, 1997). Several researchers (Knuth, 1999, 2002a, 2002b; Martin & Harel, 1989) have noted that many North American teachers, for various reasons, fail to rise to this challenge. According to Knuth, the US secondary school mathematics teachers who participated in his study had minimal knowledge about the role and function of mathematical proof within the mathematics classroom. Since the teacher's conceptions of proof inevitably influences both the role and nature of the instruction of mathematical proof within a mathematics classroom, limited knowledge in this aspect of mathematics instruction typically prompts feelings of uncertainty and a lack of confidence when it comes to teaching the concept. This may explain to a large degree why mathematical proof are either ignored, or compartmentalized as a special topic within school mathematics curriculum.

The Study

Effective mathematics teachers understand and truly *know* the mathematics that they are teaching. Further more, they are flexible in their teaching practices as they draw on that knowledge appropriately and creatively as they instruct their students (Shulman, 1986, Ball & Bass, 2000). In other words, a strong content knowledge helps them to incorporate useful representations, unifying concepts, examples and counter examples for clarification and helpful analogies (Grouws & Shultz, 1996).

NCTM (1991, 2000) considers that effective teaching requires the ability to understand what students know and need to learn and then challenge and support them to learn it well. This is infact a complex task. Hence teachers should have a profound understanding - deep, vast and thorough understanding of the mathematics involved (Ma, 1989). Knuth (2002a, 2002b) mentions that the teachers' understanding and confidence in mathematical proof influences both the role that s/he assigns to proof in her/his mathematics classrooms and her/his instructional approach in teaching. Jones (1997) notes that the teaching of mathematical proof places significant demands on both the subject matter and pedagogical knowledge of secondary mathematics teachers.

Reform initiatives by several organizations such as NCTM, Mathematical Association of America (MAA), National Research Council (NRC) have called for a new vision in teaching and learning mathematics. Hence, as Capraro et al. (2005) noted, these initiatives have led teacher preparation institutions to provide the foundation that enables neophyte instructors to grow into effective teachers. Hence, within the current context of reform, it becomes critical to know more about the student teachers' understanding of mathematical proof.

Several researchers (Ball & Wilson, 1990; Fuller, 1997; Ma, 1999; NRC, 2001) have indicated that teachers with extensive subject matter knowledge will be able to build on students' prior knowledge and will confidently promote student thinking and reasoning in the classroom. Hence, I conducted a study to assess the degree to which pre-service teachers feel confident about instructing mathematical proof. I took a case study approach in this research, collecting written response and interview data. The study involved two phases: Phase 1: Participants' responses to word based, mathematical and representational tasks; and, Phase 2: Interview of selected candidates. Jones (1997) had grounded his study in the belief that confidence level in the subject matter depends upon the subject matter knowledge of the teacher. Hence, based on Jones (1997, 2000), Fuller (1997), Even (1993), Fennema & Franke (1992), I associated high levels of confidence with high levels of conceptual understanding and, conversely, low levels of confidence with low levels of conceptual understanding. This article reports my analysis of the representational task—the concept map. I established two primary indicators for high-level understanding: first, the form and structure of the concept map produced, specifically in terms of three key features—the different forms of the maps, the number of key terms used in the maps, and the number of specified relationships among key terms and the cross-links indicated; and, second, the manner in which the construction of the map reflected a hierarchy of 'proving' as established by Balacheff (1988). His four types of proofs are 1) *naïve empiricism* in which the truth of a result is verified with a few examples; 2) *crucial experiment* in which a result is verified on a particular case which is recognized as typical; 3) *generic example* in which the truth of assertion is made explicit using a prototypical case; and 4) *thought experiment* in which operations and foundational relations of the proof are dissociated from the specific examples considered (in this case, the proofs are based on the use of and transformation of formalized symbolic expressions). The conceptual understanding displayed by means of the concept map was later mapped onto Balacheff's (1988) taxonomy of proofs.

17 pre-service teachers who participated in the study were completing the final semester of their teacher education program at a large Canadian university when this study was conducted. Data collection took place two weeks prior to the start of the final practicum (classroom teaching experience). All participants were Mathematics majors who had completed at least twelve 3-credit courses in math. The teacher education program at this University consists of a five-year combined degree. Students are required to take at least twelve 3-credit courses in their subject of major. Students with a Bachelors' degree, were able to enter into an "after-degree" program typically consisting of two years of additional study in Education. The study was conducted with the approval of University Ethics Committee.

Numerous studies related to mathematical proof undertaken with participants at different levels of schooling and from different perspectives, have been reported in the literature. Studies extend from the university –level perspective of students and teachers (Raman, 2003; Housman & Porter, 2003) to the secondary level point view of students and teachers (Balacheff, 1988; Healy & Hoyles, 2000; Knuth, 2002a, 2002b) to the perspective elementary –level student teachers (Martin & Harel, 1989). Most of the studies make use of traditional assessment tools like paper & pencil and interviews, with the exception of Jones (1997) who studied secondary-level student teachers' conceptions of mathematical proof with concept maps. With this article, I aim to contribute to reducing that gap in the literature. In this article I focus on student teachers' understanding in mathematical proof displayed through concept maps.

The Method

Noting that the limited subject knowledge of both pre-service and in-service mathematics teachers is a matter of concern, Jones (1997) undertook a small-scale investigation of United Kingdom pre-service secondary school mathematics teachers and the ways in which they conceived of mathematical proof and proving. He wanted to know just how confident student teachers felt about the prospect of teaching mathematical proof. Jones invited the group of student teacher participants to brainstorm a list of key terms that they associated with the concept of mathematical proof. The students produced a list of 24 terms. Jones then asked each student teacher to create a concept map that represented his/her understanding of mathematical proof. Students were permitted to use any or all of the key terms previously brainstormed. Using a blank sheet of paper, participants arranged the terms as each saw fit, joining terms in what each perceived as a meaningful way. Then each student indicated the relationships among the key terms by drawing lines and/or writing descriptive words on the map. Jones analyzed student maps in terms of three criteria: the specific terms used, the frequency of terms used, and the nature of the relationship (if specified) between any two terms. According to Jones, the higher the student's Grade Point Average (GPA), the more terms the student was likely to use in constructing the map. Furthermore, the student teacher with the highest GPA produced the most sophisticated map: this student added terms that were not in the original list of 24. Jones concludes that the degree of confidence a student teacher experiences, and the likelihood of his/her future success in teaching mathematical proof, depends upon the construction of sound knowledge, both in terms of subject area and pedagogy

Participants in Jones (1997) study generated the following list of 24 terms that they associated with the idea of "mathematical proof." The terms conclude are: (1) Euclidean, (2) Observation, (3) Logic, (4) General Case, (5) Trial and Improvement, (6) Theorem, (7) Graphical, (8) Assumptions, (9) Axioms, (10) Irrefutable, (11) Syllogism, (12) Deduction, (13) Definitive, (14) Implies, (15) Postulate, (16) Lemma, (17) By contradiction, (18) Explanation, (19) Hypothesis, (20) Examples, (21) Precision, (22) Proposition, (23) Reasoning, (24) Abstraction. I used the same list and gave these terms to my study group. The participants were then given sufficient time to construct a concept map of "mathematical proof" using all, some, or none of the terms provided. They could also integrate any other term that they believed was relevant to "mathematical proof".

The participants in my study, as previously noted, had already taken several university-level mathematics courses dealing with proofs and proving; therefore, it was logical to assume that they would recognize and understand most of the concepts/terms provided. The terms associated with mathematical proofs are universal, there was no reason to assume to that Canadian students would have difficulty with the terms generated by British students. Moreover, I decided to use Jones (1997) list because it would be easier to create a concept map with some familiar terms/concepts than having them generate concepts and then create a map. It was also probable that they had already developed mental models of ‘mathematical proof’ as a result of their previous exposure to ‘proof and proving’ in various mathematics courses. Hence, they could easily utilize some of the 24 terms/concepts presented to them. Rebich & Gautier (2005) noted that researchers in cognitive science have found that the knowledge learners posses is a very strong determinant of what information they attend to. They also noted that prior knowledge can be seen as the foundation for integration of new concepts. With this in mind, I thought that the list that I provide will act as scaffold that will help them integrate their ideas about the concept of proof. I did not think that providing student teachers will limit the flexibility and thinking of students as they were given the option of generating other terms, if they wish to.

The study group had discussed concept maps in another compulsory course, in a fairly detailed manner, but in a non-mathematical context. Hence I did not discuss/explain the construction of concept maps as it might pertain to the context of mathematics before the study.

Considerations Concerning Analysis

There are a number of important considerations when one analyzes a concept map. First, there is no such thing as a single correct map; rather, there will be a multitude of possible ways in which one can generate a concept map, with some maps serving as more informative representations of conceptual understanding than others (for example, maps that display labels and/or connecting verbs that make relationships explicit, and relationships that are clearly appropriate, reflect considerable conceptual understanding). Next, one may analyze a concept map in terms of the *absence* of essential concepts and the appropriateness or inappropriateness of the relationships that have been made explicit by the mapmaker with labels or connecting verbs. Finally, one may analyze a concept map according to its general form.

My analysis focuses mainly on the structure of the map, especially the degree of complexity indicated by the general form. Vanides et al. (2005) identify five typical structures for concept maps: (1) Linear, (2) Circular, (3) Hub, (4) Tree and (5) Network. The College of Agricultural, Consumer, and Environmental Sciences (CACES, n.d), at the University of Illinois, also offers four general categories of concept maps: 1) Spider Maps, 2) Hierarchy Maps, 3) Flow Chart Maps, and 4) Systems. There is a slight difference between the two categorizations as the structures are based on the forms of maps identified in their respective studies. According to CACES, concept maps that have a central theme or unifying factor that has been placed in the center, and sub- themes radiating from it, can be called a "spider map". A "Hub" map of Vanides et al (2005) classification can be considered almost equivalent to the "spider map". The type of map that presents information in descending order of importance (from top to bottom) is a hierarchy map. A concept map that organizes the concepts in a linear format is called a "flow chart" map. This map is similar to the "linear" map identified by Vanides et al. And the map that organizes information in a similar flow-chart format with the addition of "inputs" and "outputs" can be termed a "systems map". Classification systems are by no means definitive; nor are they exhaustive. A conceptual map will ultimately take whatever form best serves the cognitive needs of the individual constructing it—hence, structure/form in concept maps is always variable. To identify the structures generated by the participants of my study, I will use the structures identified by both Vanides et al. and CACES for my analysis. In other words, I will use the categorization indentified either by Vanides et al. or CACES, depending on the structure identified in my study.

More critical than the form of the map is the extent to which it illustrates complex conceptual relationships. One must analyze concept maps carefully in terms of how the key terms are used and the way in which relationships among them are specified (Jones, 1997). Concept maps that incorporate multiple ideas/concepts in ways that clarify conceptual relationships and cross-relationships (commonly referred to as network maps) demonstrate a sophisticated level of understanding (Jones, 1997; Vanides et al., 2005). Vanides et al. note that both proficient students and subject experts tend to create highly *interconnected* maps- maps that indicate relations and cross relations, while novices tend to create simple structures that are linear, circular, or organic. A network map that also includes important propositions that correctly describe the conceptual relationships that are foundational to the main ideas is evidence of high-level understanding, indeed.

Data Analysis

Three of the seventeen students did not attempt the task. Of the remaining fourteen students who did, none generated a high-level structure (that is, a complicated structure with extensive interconnectedness among concepts). Most of the maps were linear, organic, tree, systems, or spider-like structures (see Figure 2, 3, 4, and 5 as well as Table 1). In other words, all concept maps were simple and straightforward. For the most part, concepts were mapped within oval shapes and connected by lines; however, few students made these connections explicit by using arrows or labels. Only two students used propositions (verbs) to describe the relationships between the concepts.

The task of generating a concept map to demonstrate conceptual understanding of "mathematical proof" proved to be quite a challenge for the student teachers that participated in this study. I noticed that most of these student teachers somehow tried, to fit into the maps, all of the terms/concepts that they were familiar with. In most cases, the student placed the word "Proof" or "Mathematical Proof" at the center of the concept map with other concepts branching out from it. This formation represents the easiest possible concept map to construct.

Jones (1997) had used this concept map method to identify student teachers with more extensive subject knowledge of proof. He discovered that those teachers who had completed more mathematics courses and who had received higher grades were able to produce more sophisticated maps involving the use of a high number of key terms. I did notice that the student teacher who took the most mathematics courses and who referred to himself as a "mathematics geek," produced the most complex of all the concept maps. His map took the form of a tree with branches and sub-branches. He utilized all 24 concepts from the list as well as five additional terms (see Figure 3). This is consistent with Jones' (1997) findings: those who excelled at mathematics also produced the most complicated maps of mathematical concepts. I found it especially interesting that none of the participants in my study produced maps showing interconnections among terms/concepts.

Major Categories of Concept Maps

The concept maps produced by the student teachers in my study fell into four major categories: 1) Linear Maps, 2) Tree Maps, 3) Systems Maps and 4) Spider Maps. A brief discussion of some of the exemplars of each of these different types of concept maps follows below.

Linear Map

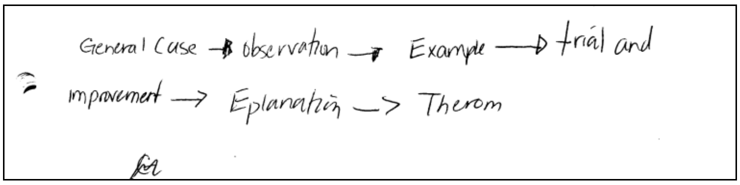

Figure 2. Linear Map

Although the CACES (n.d) identifies this type of concept map as a "Flow Chart," I prefer to call it a linear map because this term more accurately describes the map's structure. As noted before, Vanides et al. (2005) also favour the term "linear".

Based on this linear concept map, it seems that the student begins by observing from a general case—in other words, a general observation or a conjecture—and then indicates that one must try another example. If the example works, one must provide an explanation for the result. If it does not work, one must try "trial and improvement". The structure of this map suggests conceptual understanding that is in line with Balacheff's (1988) *crucial experiment.* At this level, one deals with the question of generalization by examining a case that is not very particular. If the assertion holds in the considered case, the student will argue that it is valid. In other words, at this level the thinker checks the statement with a carefully selected example that is representative of a certain class.

This mapmaker did not include in the map either the term "proof" or "mathematical proof;" rather, s/he substituted the term "explanation." This may be because the words "proof" and "mathematical proof" were not included in the list of given terms. During the interview, when enquired about why the map maker had used the term "explanation" instead of "proof", s/he had no specific reason to it. The inclusion of the "term" explanation may imply that this student teacher believes that *explanation* is one of the functions of mathematical proof. In school mathematics, one of the main purposes of introducing proof is to enhance understanding (Leddy, 2001; Hanna, 1990). However, it is interesting to note that this mapmaker, when asked to explain what a proof is, s/he did not explain proof in terms of "explanation", but in terms of "verification" and "derivation".

Tree Map

The following concept map (Figure 3) is quite close in its form to a "tree". This was the most complex concept map produced by a participant (As noted earlier, it was the work of the student who self-identified as a "mathematical geek"). Yet despite its relative complexity, the map lacks the verbs that would link concepts. This student teacher has laid out all of the different concepts very carefully. This concept map also illustrate has all of the characteristics of a *hierarchical* map, where general terms are placed closest to the word "proof" and specific terms furthest away. Furthermore, all terms/concepts are appropriately placed within the map, suggesting that the student teacher has a *holistic* understanding of the concept. This holistic presentation of the concepts as well as the extensive detail of the map suggests that this student's understanding of mathematical proof operates at the level of a thought experiment.

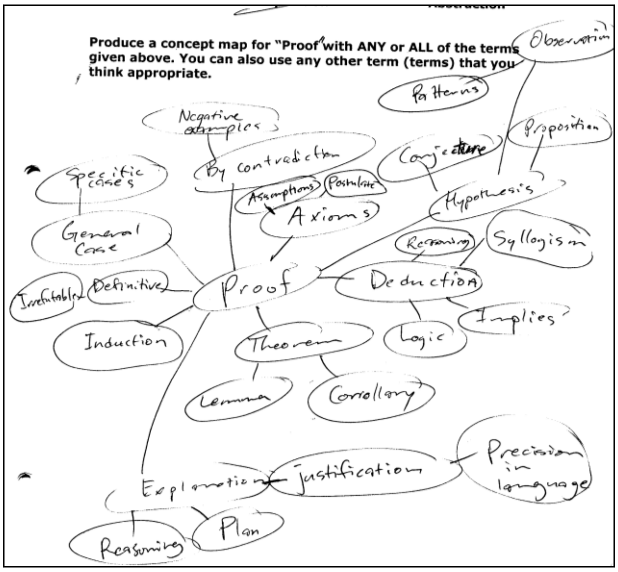

Figure 3. Tree Map

This observation is consistent in this student teacher's written task and also interview. As noted before, when students operate at the thought experiment level they are able to distance themselves from action and make logical deductions based upon an awareness of the properties and the relationships. It is at this, the fourth and the highest level in Balacheff's hierarchy of proof that students move from practical to intellectual proofs. Balacheff (1988) called this level 'conceptual justification'. At this level, actions are internalized and dissociated from the specific examples considered. The justification is based on the use of and transformation of formalized symbolic expressions.

Systems Map

The following concept map (Figure 4) takes the form of "Systems Map". The inputs are "Hypothesis", "Trial and Improvement" and "Theorem" and the Output is "irrefutable proof." The message conveyed by this concept map can be paraphrased as follows: Based on the "hypothesis" and using "trial and error" (or examples) a theorem can be formulated. This theorem can be proved by using a "Direct Method" or by using an "Indirect Method". Whether one uses a "direct method" or an "indirect method," *axioms, reasoning,* and *logic* will play a major role in the "proving" or the "disproving". The concept map provides one major insight into this student teacher's conception of mathematical proof: s/he believes that *once a theorem is proved, it is irrefutable*. The belief that *once a theorem is proved, it is irrefutabl*e suggests an absolutist philosophy of mathematics (Ernest, 1990). Even though far fewer concepts are evident in this map than in the more sophisticated 'tree' map, the concepts are deliberately placed

The input section includes the phrase "trial and error" (that is, trying out various examples); but the student teacher also includes the words "general case", which is not quite relevant in the set of concepts s/he used. From the rest of the concepts used in the map, it can be seen that the student teacher know the steps involved in "proving". Research indicates that the greater the number of terms in the map and the interconnectedness between them, the deeper the conceptual understanding of the mapmaker.

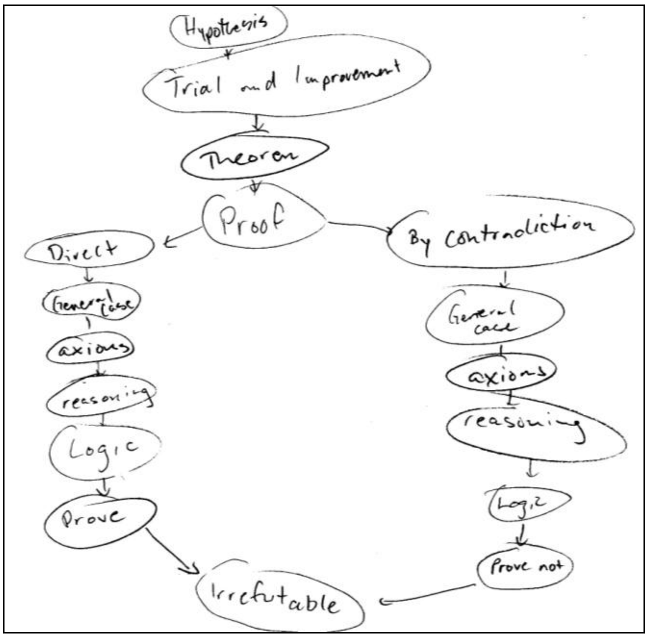

Figure 4. Systems Map

Spider Maps

The most popular format for constructing a concept map proved to be the "spider maps". Here, the mapmaker placed the central theme or unifying factor in the center of the map and then organized related items around the center (see Figure 5). In my view, this sort of mapping does not require high-level cognition. Any number of terms/concepts can be placed around a main concept in no time at all. Vanides et al., (2005) consider “Linear Maps” (Figure 2) to be the most simple of all the different forms of concept maps; however, some level of thinking is essential even at this level of mapping since the mapmaker needs to determine where to place each concept within a linear relationship. In constructing a “spider map”, a student places any number of concepts around a given term. Relationships need not be as carefully considered as they are when constructing even a simple linear map. “Spider maps”, clearly, do not require the level of thinking that is needed when constructing “linear maps”. It would be safe, then, to align the level of understanding of mathematical proof as displayed within this concept map with one of the lowest levels in Balacheff’s (1988) hierarchy of proofs.

Having said that, however, I do include one example of a spider map in which it seems apparent that the mapmaker gave some thought to the relationships among the items in the circle (see Figure 6). As would be expected with a spider map structure, the central concept, “proof”, has been placed in the center of the map with related terms around it in a circle. The mapmaker uses various linking verbs to connect the terms/concepts that s/he has selected. Unlike other examples of the spider map structure, however, here the mapmaker employs a clockwise direction to assist in explaining what a proof is. I infer his/her thinking process from the concept map as follows: Proof always starts off with “hypothesis/ abstraction/ assumption, proposition”. Proof is accomplished by “logic”; it uses “theorems”, “axioms”, “examples” and “reasoning”. Proofs can be proved either by “deduction” or by “contradiction”. Proofs require “precision”. A process that could be used to improve “proof” is “trial and error”. Proofs can be learned by “observation”. Proofs should apply to the “general case” and should be “definitive”. “Nice” proofs are obtained by “Euclidean” and “graphical” methods. Proofs end with “postulates” and “explanation”. This example demonstrates that even with a relatively simple solar system form, a mapmaker can generate a sophisticated map as long as he/she possesses a deep understanding of the concept. I suggest that this mapmaker displays good understanding of the processes involved in proving. However, when it came to actually proving the tasks, I noticed that this particular student did not display his/her proficiency. Only two student teachers used linking terms to connect the main concepts with the others. For example, Figure 6 is one of them.

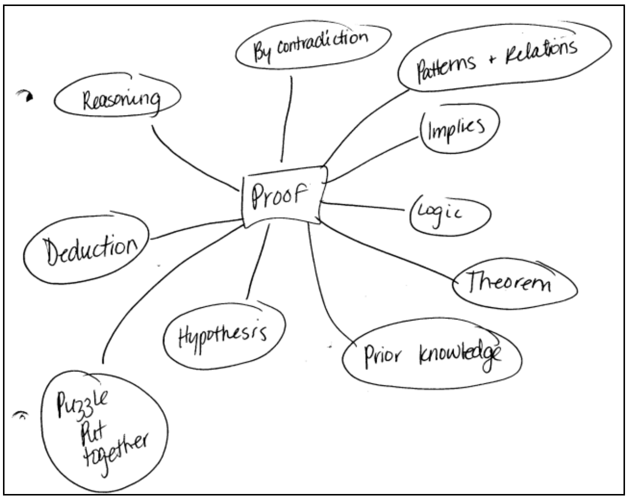

Figure 5. Spider Map

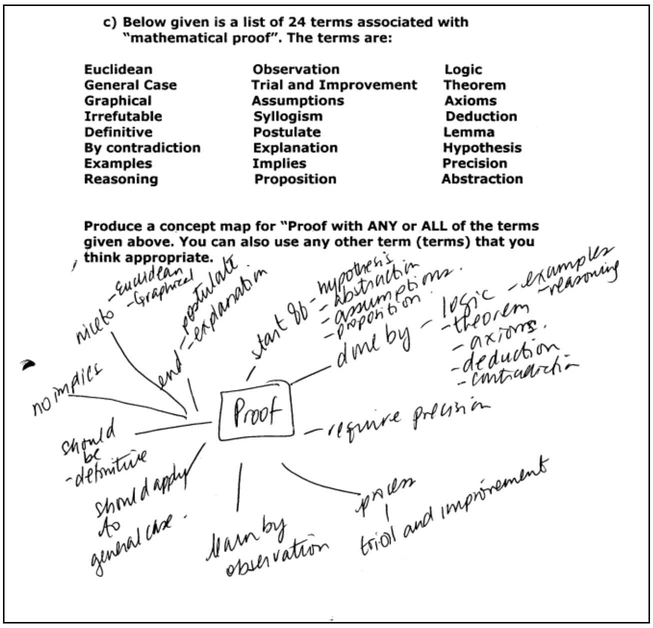

Figure 6. Another Example of a Spider Map

General Comments

The table given below (Table 1) provides a summary of the different forms of concept maps generated by the students as well as the number of key terms used and the relationships specified. It can easily be seen that among the different styles of maps generated, the "spider map" was the most popular. I believe that this was because this format best accommodated all of the given terms. As noted earlier, only two student teachers used verbs to specify the relationships.

Table 1
Observations from Concept Maps

Name	Form/ Shape of the Map	# of key terms used	Specific Relationship shown (Yes/No)	# of cross links between concepts	Terms outside the list used
A	Spider	19	No	0	Proof
B	Tree	21	Yes (by arrow)	0	Proof
C	-	-	-	-	-
D	Spider	19	No	0	Proof
E	Spider	11	No	0	Proof, prior knowledge patterns & relations puzzle put together
F	Not Identifiable	19	Once	0	Proof
G	Linear	19	Yes (arrows)	0	Problem, grouping, theory, special case
H	Linear	7	Yes (arrows)	0	-
I	Tree	13	No	0	Proof
J	Spider	11	Yes (verbs)	0	Proof
K	Tree	29	0	0	Proof, specific cases, negative examples, plan, induction
L	Systems	17	Yes (arrows)	0	Proof
M	Not Identifiable	10	Yes (verbs)	0	-

N	Spider	24	-	0	Proof
O	-	-	-	-	-
P	Systems	8	-		-
Q	-	-	-	-	-

Of the terms added by the mapmakers (that is, terms that were not included in the original list of 24), the most common was “proof” or “mathematical proof”, which is quite understandable. Most of the student teachers failed to use a high number of the original 24 terms in any logical manner. Those who used most of the 24 terms had opted for the “spider” format, which, by virtue of its visual design, easily accommodates quite a number of terms. If one uses this criteria—a high number of terms *used in a deliberate and logical way—* as an indication of high confidence levels in understanding and teaching mathematical proof, then this group cannot be categorized as a group of student teachers who are confident in their understanding of and future teaching of “mathematical proof”. Using Balacheff’s (1988) terminology, most of the student teachers in this group operated at a *pragmatic* justification level. The category of pragmatic justifications includes the first three levels in Balacheff’s proof scheme: naïve empiricism, the crucial experiment and the generic example. Due to their exposure to different mathematical courses and their familiarity with most of the given terms, this one could notice traces of thought experiments in all of their proofs.

Conclusion

In at least one of the compulsory education courses in their program of studies, these pre-service teachers had been introduced and discussed fairly in detail about concept maps. Yet, when asked to generate maps in relation to mathematics, they had difficulty generating sophisticated representations of learning; that is, they could not effectively incorporate the different ideas and concepts that they had learned about “proofs and proving” (the terms provided) into sophisticated visual representations of their mathematical understanding.

The written mathematical tasks that I administered along with this concept map task indicated that many of the student teachers had difficulty with secondary-school level mathematical proof concepts. If the concept maps that these students generated are to be taken as triangulating information to students’ level of thinking as it pertains to mathematical proof, then the implications are quite serious. As noted earlier, current reform in mathematics education and curriculum calls for an increased emphasis on “reasoning and proof” as key stepping-stones towards a better understanding of mathematics. The high school student’s understanding of mathematics, and his/her ability to solve problems and develop logical reasoning and justification skills, are shaped by the teaching that s/he encounters in schools

(NCTM, 2000). Hence, teachers themselves should have a healthy understanding of mathematical proof. Galbraith (1982) expressed concern about the quality of an educational system in which students who fail to master essential mathematical concepts later return to the system as mathematics teachers. He referred to this as a "recycling effect" when teachers, who lack fundamental skills, fail to teach students those skills, and those students, becoming teachers themselves, then perpetuate the process. A "recycling effect" seems very likely if student teachers who demonstrate an inadequate understanding of mathematical proof and reasoning later return to the educational system as mathematics teachers faced with the specific challenge of teaching proof.

Wu (1997) notes that, student difficulty with mathematical proof warrants a re-examination of students' university-level mathematics courses. He suggests that the undergraduate mathematics education is built on "Intellectual –Trickle Down Theory" (p. 5). Professors direct their teaching towards the best students while believing that, somehow, the rest will take care of themselves. He also notes another aspect of undergraduate mathematics education, what he calls "delayed gratification" (p. 4), where the instructors believe that if students don't understand something in a particular course now, they will surely come to understand it when they do their graduate study or when they begin research. However, in reality only 20% or less of the math graduates continue with graduate work. For the vast majority of undergraduates, these courses are the "grand finale of their mathematical experience" (p. 4). Those who opt to become secondary mathematics teachers will be among this vast majority of students who were fed "technicality after technicality" in their mathematics courses, and did not understand most of the material that was taught. A Nation at Risk, document published by US Department of Education in 1983, indicates that too many teachers are drawn from the bottom quarter of those students who graduate from university/college. Wu insists that the "technical inadequacy of mathematics teachers" (p. 8) is a serious problem, and that it is high time that mathematicians and mathematics educators sit together and design "elementary mathematics courses with an advanced point of view". Jones (2000) has expressed a similar concern about student teachers' lack of subject content knowledge. My study points towards a need for further research in this area. Further in-depth studies of student teachers' understanding of mathematical content areas other than "mathematical proof" are necessary if teacher education programs are to become more effective at educating future mathematics teachers.

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